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# PERFECT MATCHINGS IN BALANCED HYPERGRAPHS — A COMBINATORIAL APPROACH

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Our topic is an extension of the following classical result of Hall to hypergraphs: A bipartite graph $G$ contains a perfect matching if and only if for each independent set $X$ of vertices, at least $|X|$ vertices of $G$ are adjacent to some vertex of $X$. Berge generalized the concept of bipartite graphs to hypergraphs by defining a hypergraph $G$ to be balanced if each odd cycle in $G$ has an edge containing at least three vertices of the cycle. Based on this concept, Conforti, Cornuéjols, Kapoor, and Vušković extended Hall's result by proving that a balanced hypergraph $G$ contains a perfect matching if and only if for any disjoint sets $A$ and $B$ of vertices with $|A|>|B|$, there is an edge in $G$ containing more vertices in $A$ than in $B$ (for graphs, the latter condition is equivalent to the latter one in Hall's result). Their proof is non-combinatorial and highly based on the theory of linear programming. In the present paper, we give an elementary combinatorial proof.

Throughout this paper, a *hypergraph* is a pair $G=(V,E)$ where $V(G):=V$ is a finite set and $E(G):=E$ is a set of subsets of $V$. As usual, the elements of $V$ and $E$ are called *vertices* and *edges* respectively. A sequence $P=v_0f_1v_1f_2v_2\ldots f_\ell v_\ell$ of vertices $v_0,v_1,\ldots,v_\ell$ and edges $f_1,\ldots,f_\ell$ with $v_{i-1},v_i\in f_i$ for all $i\in\{1,\ldots,\ell\}$ is called a *path* if $v_0,v_1,\ldots,v_\ell$ are pairwise distinct. $P$ is called a *cycle* if $\ell\geq 3$, if $v_0,v_1,\ldots,v_{\ell-1}$ are pairwise distinct, and if $v_\ell=v_0$. In both cases, $\ell$ is called the *length* of $P$ and we define $V(P):=\{v_0,v_1,\ldots,v_\ell\}$ and $E(P):=\{f_1,\ldots,f_\ell\}$. Moreover, we call $P$ *strong* if $f_i$ has only $v_{i-1}$ and $v_i$ in common with $V(P)$ for each $i\in\{1,\ldots,\ell\}$ (particularly, $f_1,\ldots,f_\ell$ are pairwise distinct).

For each $M\subseteq E$, we define $V(M):=\{x\in V;\ x\in f \text{ for some } f\in M\}$. $M$ is called a *matching* if $M$ consists of pairwise disjoint edges. If additionally,

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$V(M)=V$ (i.e. $M$ covers each vertex), then $M$ is called *perfect*. There are several extensions of the following classical result of Hall [5] about perfect matchings in bipartite graphs to hypergraphs (for the graph theoretic terminology, the reader is referred to Bollobás [2]):

**Theorem 1.** *A bipartite graph $G$ has a perfect matching if and only if for each independent set $X$ of vertices (i.e. no edge of $G$ connects two vertices of $X$), at least $|X|$ vertices of $G$ are adjacent to a vertex of $X$.*

With regard to the fact that a graph is bipartite if and only if it contains no cycle of odd length, Berge [1] defined a hypergraph to be *balanced* if it does not contain strong cycles of odd length (note that paths and cycles in graphs are always strong). Conforti, Cornuéjols, Kapoor, and Vušković [3] extended also the latter condition in Theorem 1 to hypergraphs: A hypergraph $G=(V,E)$ satisfies the *Hall-condition* if for all disjoint $A,B\subseteq V$ with $|A|>|B|$, there exists an edge in $G$ containing more vertices in $A$ than in $B$ (it is easy to see that for graphs, this condition is indeed equivalent to the latter one in Theorem 1). These two concepts lead to the following generalization of Theorem 1 which was proved in [3].

**Theorem 2.** *A balanced hypergraph $G$ has a perfect matching if and only if $G$ satisfies the Hall-condition.*

The proof in [3] is non-combinatorial and highly based on the theory of linear programming. Here we present an elementary combinatorial proof.

To prepare the proof, we continue by providing some technical results about matchings in arbitrary not necessarily balanced hypergraphs. In the following, $G=(V,E)$ is always a hypergraph.

(1) Let $S\subseteq V$ be nonempty and assume that for each $x\in S$, there exists a matching $M_x$ of $G$ with $V\setminus S\subseteq V(M_x)$ and $x\notin V(M_x)$. Then (i) or (ii) hold.
  - (i) $G$ contains a strong path $P=v_0f_1v_1\dots f_\ell v_\ell$ of positive even length such that $V(P)\cap S=\{v_0,v_\ell\}$.
  - (ii) $G$ contains a matching $M$ with $V(M)=V\setminus S$.

**Proof.** By the premises of (1), $G$ contains a matching $M$ with $V\setminus S\subseteq V(M)$ such that $|V(M)|$ is as small as possible. We may assume that there exists some $v\in V(M)\cap S$ (otherwise (ii) holds). Construct a directed graph $H$ as follows: Let $A_1:=M\setminus M_v$ and $A_2:=M_v\setminus M$ and define the vertex-set $V(H)$ of $H$ to be the disjoint union $A_1\cup A_2$. Then for any distinct $f,g\in V(H)$, take an edge $fg$ directed from $f$ towards $g$ and define the edge-set $E(H)$ of

$H$ to be

$$\{fg;\ f\in A_1, g\in A_2, f\cap g\neq\emptyset\}$$
$$\cup\,\{fg;\ f\in A_2, g\in A_1, (f\cap g)\setminus S\neq\emptyset\}.$$

Additionally, we define

$$B_1 := \{f\in A_1;\ v\in f\},$$
$$B_2 := \{f\in A_2;\ f\setminus V(M)\neq\emptyset\}.$$

We prove that $H$ contains a directed path from a vertex of $B_2$ to a vertex of $B_1$. Define $C$ to be the set of all vertices $f$ of $H$ such that $H$ contains a directed path from a vertex of $B_2$ to $f$ and suppose for contradiction that $B_1\subseteq\overline{C}:=V(H)\setminus C$, Clearly, we have $B_2\subseteq C$ and $H$ does not contain edges directed from a vertex of $C$ towards a vertex of $\overline{C}$. Define $M'$ to be the disjoint union

$$(A_1\cap C)\ \cup\ (A_2\cap\overline{C})\ \cup\ (M\cap M_v).$$

$M'$ is a matching of $G$ since $M$ and $M_v$ are matchings and since for each $f\in A_1\cap C$ and for each $g\in A_2\cap\overline{C}$, we have $fg\notin E(H)$ and thus $f\cap g=\emptyset$.

We prove $V\setminus S\subseteq V(M')$. Let $x\in V\setminus S$. Then since $V\setminus S\subseteq V(M)\cap V(M_v)$, there exist $f\in M$ and $g\in M_v$ with $x\in f\cap g$. We may assume that $f\in A_1$ and $g\in A_2$ since otherwise $x\in V(M\cap M_v)\subseteq V(M')$. Thus $fg, gf\in E(H)$ (since $x\in(f\cap g)\setminus S$) so that we must have $f,g\in C$ or $f,g\in\overline{C}$. Hence $x\in V(A_1\cap C)\cup V(A_2\cap\overline{C})\subseteq V(M')$.

Now by the minimality of $|V(M)|$ and since $v\in V(M)$, there exists some $f\in M'$ with $v\in f$ or $f\setminus V(M)\neq\emptyset$. In the first case, we have $f\in A_1$ (since $v\notin V(M_v)$) and thus $f\in B_1\subseteq\overline{C}$, contradicting the definition of $M'$. In the latter case, we have $f\in A_2$ and thus $f\in B_2\subseteq C$, again a contradiction.

Now we have proved that $H$ contains a directed path $Q$ from a vertex of $B_2$ to a vertex of $B_1$. Let $f_1, f_2, \ldots, f_k$ be the vertex-sequence of $Q$. Then since all edges of $H$ are between $A_1$ and $A_2$ and since $f_1\in B_2$ and $f_k\in B_1$, we see that $k$ is even, $f_1, f_3, \ldots, f_{k-1}\in A_2$, and $f_2, f_4, \ldots, f_k\in A_1$. Moreover, there exists some $v_0\in f_1\setminus V(M)$ and we have $v_k:=v\in f_k$. Finally, for each $i\in\{1,\ldots,k-1\}$, $f_i\cap f_{i+1}\neq\emptyset$ holds (since $f_if_{i+1}\in E(H)$) and thus we may choose some $v_i\in f_i\cap f_{i+1}$ such that $v_i\notin S$ if $(f_i\cap f_{i+1})\setminus S\neq\emptyset$. Note that $v_0\notin f_2\cup f_4\cup\ldots\cup f_k$ (since $v_0\notin V(M)$) and $v_k\notin f_1\cup f_3\cup\ldots\cup f_{k-1}$ (since $v_k\notin V(M_v)$) and that both sequences $f_1, f_3, \ldots, f_{k-1}$ and $f_2, f_4, \ldots, f_k$ consist of pairwise disjoint edges. Now we can see that $v_0f_1v_1f_2v_2\ldots f_kv_k$ is a strong path in $G$.

Since $v_k\in V(M)\cap S$, we obtain a minimal index $\ell\in\{1,\ldots,k\}$ with $v_\ell\in S$. We also have $v_0\in S$ (since $v_0\in f_1\setminus V(M)$ and $V\setminus S\subseteq V(M)$) so that

$P := v_0 f_1 v_1 \ldots f_\ell v_\ell$ is a strong path in $G$ with $V(P) \cap S = \{v_0, v_\ell\}$. Suppose that $\ell$ is odd (otherwise we have (i)). Then $0 < \ell < k$, $f_\ell \in A_2$, $f_{\ell+1} \in A_1$, and $f_\ell f_{\ell+1} \in E(H)$. Hence $(f_\ell \cap f_{\ell+1}) \setminus S \neq \emptyset$ and thus $v_\ell \notin S$ by the choice of $v_\ell$, a contradiction. ∎

We call another hypergraph $G' = (V', E')$ a *subhypergraph* of $G$ if $V' \subseteq V$ and if for each $f' \in E'$, there exists $f \in E$ with $f' = f \cap V'$. $f$ is called an *extension* of $f'$ (note that $f'$ may have more than one extension in $G$ ). Moreover, if $M' \subseteq E'$ and if $M \subseteq E$ is obtained by extending the edges of $M'$ (i.e. we replace each $f' \in M'$ by an extension of $f'$), then $M$ is called an *extension* of $M'$. Finally, if $P' = v_0 f_1' v_1 \ldots f_\ell' v_\ell$ is a path or a cycle in $G'$, then by extending $f_1', \ldots, f_\ell'$, we obtain a path or a cycle $P = v_0 f_1 v_1 \ldots f_\ell v_\ell$ in $G$ respectively. $P$ is called an *extension* of $P'$. Note that $P'$ is strong if and only if $P$ is strong. Thus subhypergraphs of balanced hypergraphs are again balanced.

Let $X \subseteq V$. Then a subset $M \subseteq E$ is called an *$X$-matching* if $f \cap g \cap X = \emptyset$ for any distinct $f, g \in M$. If additionally, $X \subseteq V(M)$, then $M$ is called a *perfect $X$-matching*. Note that $|X| = \sum_{f \in M} |f \cap X|$ in this case.

Let $A, B \subseteq V$. An edge $f \in E$ is called *$A,B$-leftdominating* if $|f \cap A| > |f \cap B|$. We call the pair $(A, B)$ *critical* if $A \cap B = \emptyset$ and $|A| > |B|$, if there is no $A,B$-leftdominating edge in $G$, and if $|A \cup B|$ is as small as possible with respect to these properties. Note that $|A| = |B| + 1$ in this case and that $G$ satisfies the Hall-condition if and only if $G$ does not contain critical pairs.

(2) Let $(A, B)$ be a critical pair of $G$, let $S \subseteq A$ and $T \subseteq B$ with $|S| > |T|$, and let $C \subseteq V \setminus (A \cup B)$. Moreover, assume that for each $x \in S$, there exists a perfect $(A \cup B \cup C) \setminus \{x\}$-matching in $G$. Then (i) or (ii) hold.
   (i) $G$ contains a strong path $P = v_0 f_1 v_1 \ldots f_\ell v_\ell$ of positive even length such that $V(P) \subseteq A \cup B \cup C$, $V(P) \cap S = \{v_0, v_\ell\}$, and $V(P) \cap T = \emptyset$.
   (ii) $|S| = |T| + 1$ and $G$ contains a perfect $(A \cup B \cup C) \setminus (S \cup T)$-matching $M$ with $V(M) \cap (S \cup T) = \emptyset$.

**Proof.** Define

$$X := (A \cup B \cup C) \setminus T$$

and for each $x \in S$, let $M_x$ be a perfect $(A \cup B \cup C) \setminus \{x\}$-matching of $G$ and

$$M_x' := \{ f \cap X;\ f \in M_x \}.$$

Moreover, define

$$G' := (X, \bigcup_{x \in S} M_x').$$

Then $G'$ is a subhypergraph of $G$. Furthermore, for each $x \in S$, since $|f \cap A| \leq |f \cap B|$ for all $f \in M_x$ and $|A \setminus \{x\}| = |B|$, we can see that $x \notin V(M_x)$ and that $|f \cap A| = |f \cap B|$ for all $f \in M_x$. Hence $M'_x$ is always a matching of $G'$ with $X \setminus S \subseteq V(M'_x)$ and $x \notin V(M'_x)$. Thus by (1), $G'$ contains a strong path $P' = v_0 f'_1 v_1 \dots f'_\ell v_\ell$ of positive even length with $V(P') \cap S = \{v_0, v_\ell\}$ or $G'$ has a matching $M'$ with $V(M') = X \setminus S$. In the first case, any extension $P$ of $P'$ in $G$ satisfies (i).

Now assume the latter case. We prove (ii). Clearly, $G$ contains an extension $M$ of $M'$ with $M \subseteq \bigcup_{x \in S} M_x$. Hence $M$ is a perfect $X \setminus S$-matching and for each $f \in M$, we have $|f \cap A| = |f \cap B|$ and $f' := f \cap X \in M'$ so that $f \cap S = f' \cap S = \emptyset$. Using all these properties of $M$, we can conclude that $V(M) \cap S = \emptyset$ and $|A \setminus S| = |A \cap V(M)| \geq |B \cap V(M)| \geq |B \setminus T|$. Hence since $|A| = |B| + 1$ and $|S| > |T|$, we obtain $|S| = |T| + 1$ and $|B \cap V(M)| = |B \setminus T|$. Thus finally, $V(M) \cap T = \emptyset$. ∎

Now we prove Theorem 2. Obviously, each balanced hypergraph with a perfect matching satisfies the Hall-condition (even the non-balanced ones). For the converse, suppose for contradiction that $G$ is balanced, satisfies the Hall-condition, and does not contain a perfect matching. We may assume that $|V| + |E|$ is minimal with respect to these properties.

(3) For each proper subset $X$ of $V$, $G$ contains a perfect $X$-matching.

**Proof.** Obviously, $G' := (X, \{f \cap X; f \in E\})$ is a subhypergraph of $G$ satisfying the Hall-condition. Hence by the minimality of $|V| + |E|$, $G'$ has a perfect matching $M'$. Clearly, each extension of $M'$ in $G$ is a perfect $X$-matching of $G$. ∎

We define $R$ to be the set of all vertices $x$ of $G$ such that $G$ contains a matching $M$ with $V(M) = V \setminus \{x\}$. Moreover, for each $f \in E$, we define $G - f := (V, E \setminus \{f\})$.

(4) For each $f \in E$, $G - f$ contains a critical pair $(A, B)$ and for each such pair, we have the following.
  (4.1) $|f \cap A| = |f \cap B| + 1$.
  (4.2) $f \cap A = f \cap R$.
  (4.3) $|f \cap A| \geq |f \setminus A|$.

**Proof.** $G$–$f$ does not satisfy the Hall-condition (otherwise by the minimality of $|V| + |E|$, $G - f$ has perfect matching which is also a perfect matching of $G$, a contradiction). Hence $G - f$ contains a critical pair. Let $(A, B)$ be any critical pair of $G - f$. Then $f$ is the only $A, B$-leftdominating edge of $G$.

To prove (4.1), we first show that for each $x \in f \cap A$, $G - f$ contains a perfect $(A \cup B) \setminus \{x\}$-matching $M_x$. Define

$$G'_x := ((A \cup B) \setminus \{x\},\ \{g \cap ((A \cup B) \setminus \{x\});\ g \in E \setminus \{f\}\}).$$

$G'_x$ is a subhypergraph of $G - f$. Moreover, since $(A,B)$ is critical in $G - f$, we can see that $G'_x$ satisfies the Hall-condition and hence by the minimality of $|V| + |E|$, $G'_x$ contains a perfect matching $M'_x$. Each extension $M_x$ of $M'_x$ in $G - f$ is as required.

To finish the proof of (4.1), suppose that $|f \cap A| \geq |f \cap B| + 2$. Then by (2) with $S := f \cap A$, $T := f \cap B$, and $C := \emptyset$, we obtain a strong path $P = v_0 f_1 v_1 \ldots f_\ell v_\ell$ of positive even length in $G - f$ such that $V(P) \cap f = \{v_0, v_\ell\}$. But $P$ together with $f$ yields a strong cycle in $G$ of odd length $\ell + 1$, a contradiction.

To prove (4.2), we first show $f \cap A \subseteq R$. Let $x \in f \cap A$. Then by (3), $G$ contains a perfect $V \setminus \{x\}$-matching $M$. If $x \notin V(M)$, then $M$ is a matching in $G$ with $V(M) = V \setminus \{x\}$ and thus $x \in R$. Suppose for contradiction that $x \in V(M)$. Since $G$ has no perfect matching, $x$ is contained in at least two edges of $M$. But now by (4.1) and since $|g \cap A| \leq |g \cap B|$ for all $g \in M \setminus \{f\}$, we obtain $|A| + 1 \leq \sum_{g \in M} |g \cap A| \leq \sum_{g \in M} |g \cap B| < |B| + 1 = |A|$, a contradiction.

To finish the proof of (4.2), suppose that there exists some $x \in (f \cap R) \setminus A$. Take a matching $M$ of $G$ with $V(M) = V \setminus \{x\}$. Then $M$ contains an $A,B$-leftdominating edge $g$. Hence $g = f$ and thus $x \in V(M)$, a contradiction.

To prove (4.3), we define

$$G' \ := \ (\ V \setminus f,\ \{g \in E;\ g \cap f = \emptyset\}\ ).$$

$G'$ is a subhypergraph of $G$ and does not satisfy the Hall-condition (otherwise by the minimality of $|V| + |E|$, $G'$ contains a perfect matching which together with $f$ yields a perfect matching of $G$, a contradiction). Take a critical pair $(A', B')$ of $G'$.

We prove that there exists $z \in f$ such that $z$ is contained in every $A', B'$-leftdominating edge $g$ of $G$. Note that $g \cap f \neq \emptyset$ for each such $g$. Hence it suffices to prove that for any two $A', B'$-leftdominating $g, h \in E$, we have $g \cap f \subseteq h \cap f$ or $h \cap f \subseteq g \cap f$. Suppose for contradiction that there exist some $u \in (g \cap f) \setminus h$ and $v \in (h \cap f) \setminus g$. Then $(g \cap h) \setminus f = \emptyset$ since otherwise we obtain a strong cycle of length three in $G$ containing $u, v$ and $f, g, h$, a contradiction. Hence $|(g \cup h) \cap A'| \geq |(g \cup h) \cap B'| + 2$. Now similarly to the proof of (4.1), we obtain a perfect $(A' \cup B') \setminus \{x\}$-matching in $G'$ for each $x \in (g \cup h) \cap A'$ so that by (2) with $S := (g \cup h) \cap A'$, $T := (g \cup h) \cap B'$, and $C := \emptyset$, there exists a strong path $P = v_0 f_1 v_1 \ldots f_\ell v_\ell$ of positive even length in $G'$ with $(g \cup h) \cap V(P) = \{v_0, v_\ell\}$. Depending on the positions of $v_0$ and $v_\ell$, $P$ together with $g$, with $h$, or with $u, v$ and $f, g, h$ yields a strong cycle in $G$ of odd length $\ell+1$ or $\ell+3$, a contradiction.

To finish the proof of (4.3), take a vertex $z \in f$ as described above and suppose that $|f \cap A| < |f \setminus A|$. Then since $|f \cap A| > |f \cap B|$, we obtain some $x \in f \setminus (A \cup B \cup \{z\})$. By (3), $G$ contains a perfect $V \setminus \{x\}$-matching $M$. Since $x \notin A \cup B \cup A' \cup B'$, $M$ has an $A, B$-leftdominating edge $g$ and an $A', B'$-leftdominating edge $h$. We have $g = f$ (recall that $f$ is the only $A, B$-leftdominating edge in $G$), $g \neq h$ (since $h$ contains vertices in $A'$ contrary to $f$), and $z \in h$. Thus $z \in g \cap h \cap (V \setminus \{x\})$, a contradiction. ∎

Note that $V \neq \emptyset$ (since $G$ has no perfect matching) and that each $x \in V$ is contained in some $f \in E$ (to see this, apply the Hall-condition to $\{x\}$ and $\emptyset$). Hence $E \neq \emptyset$ and thus by (4.1) and (4.2), $R \neq \emptyset$. Now by taking some $x \in R$ and by applying (4.2) and (4.3) to the edges of a matching $M$ of $G$ with $V(M) = V \setminus \{x\}$, we see that $|R| > |V \setminus R|$. Hence by the Hall-condition, $G$ contains an $R, V \setminus R$-leftdominating edge $f$. By (4.1), (4.2) and (4.3), we can see that $f \subseteq A \cup B$ for some arbitrary critical pair $(A, B)$ of $G - f$. Moreover, for each $x \in f \cap A$, there exists a matching $M_x$ of $G$ with $V(M_x) = V \setminus \{x\}$. Note that we always have $f \notin M_x$ so that $M_x$ is also a perfect $V \setminus \{x\}$-matching of $G - f$. Now by (2) with $S := f \cap A$, $T := f \cap B$, and $C := V \setminus (A \cup B)$, we obtain a strong path $P = v_0 f_1 v_1 \dots f_\ell v_\ell$ of positive even length in $G - f$ with $V(P) \cap f = \{v_0, v_\ell\}$ or there exists a matching $M$ of $G - f$ with $V(M) = V \setminus f$. In the first case, $P$ together with $f$ yields a strong cycle in $G$ of odd length $\ell+1$, a contradiction. In the latter case, $M \cup \{f\}$ is a perfect matching of $G$, again a contradiction. — Now we have finished the proof of Theorem 2.

**Remarks.**

(i) Note that (1) and (2) are valid even if $G$ is not balanced. They could perhaps be used to extend Theorem 2 to a larger class of hypergraphs.

(ii) It seems natural to look for a "deficiency-version" of Theorem 2. Note that if $G$ contains a matching covering all but at most $d$ of the vertices, then for any disjoint subsets $A, B \subseteq V$ with $|A| > |B| + d$, there is an $(A, B)$-leftdominating edge in $G$. If $G$ is a bipartite graph, then the maximum value $d$ for which $G$ satisfies the latter condition equals the minimum cardinality of $V \setminus V(M)$ where $M$ is a matching of $G$ (see Ore [6]). However, this is not true in general for balanced hypergraphs as can be seen by the following trivial example: Define

$$V := \{1, 2, \dots, 2n+1\} \quad (n \geq 2),$$
$$f := \{1, 2, \dots, n+1\},$$

$$g := \{n+1, n+2, \ldots, 2n+1\},$$
$$G := (V, \{f, g\}).$$

It is easy to check that for any disjoint $A, B \subseteq V$ with $|A| > |B|+1$, $G$ has an $A,B$-leftdominating edge and that each matching of $G$ leaves at least $n$ vertices uncovered.

The duality theory in Fulkerson, Hoffman and Oppenheim [4] yields the following: It is possible to cover all but at most $d$ vertices by a matching if and only if each function $w$ from $V$ to $\{0,1,2,\ldots\}$ with $\sum_{v\in f} w(v) \geq |f|$ for each edge $f \in E$ has total weight at least $|V|-d$, i.e., $\sum_{v\in V} w(v) \geq |V|-d$.

(iii) It is a challenging task to find a combinatorial matching algorithm for balanced hypergraphs.

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